Journal of Advances in Mathematics Vol 15 (2018) ISSN: 2347-1921 https://cirworld.com/index.php/jam

# On Twisted Sums of Sequence Spaces.

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**Abstract**
We prove the existence of non trivial twisted sums involving the p$^{th}$ James space $J_p(1 \leq p < \infty)$, the Johnson-Lindenstrauss space $JL$, the James tree space $JT$, the Tsirelson's space $T$ and the Argyros and Deliyanni space $AD$. We also present non trivial twisted sums involving their duals and biduals. We show that there are strictly singular quasi-linear maps from the spaces $T, T^*, AD$ and $JT$ into $C[0,1]$. We discuss the Pelczynski's property$(u)$ for the twisted sums involving these spaces which extends a p$^{\text{th}}$James-Schreier spaces $V_p(1 < p < \infty)$ or $J_2$.

**Keywords:** Twisted sums, James spaces, Tsirlson's spaces, strictly singular.
**Subject Classification:** 46B03; 46B20; 46B45.

## Introduction

A quasi-Banach space $X$ is said to be a twisted sum of two Banach spaces $Y$ and $U$ if it contains a subspace $A$ isomorphic to $Y$ and the quotient $X/A$ is isomorphic to $U$. Identifying $A$ with $Y$ we have the following short exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow U \longrightarrow 0$$

Two exact sequences $0 \longrightarrow Y \longrightarrow X_1 \longrightarrow U \longrightarrow 0$ and $0 \longrightarrow Y \longrightarrow X_2 \longrightarrow U \longrightarrow 0$ are equivalent if there is a bounded linear operator $T$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & U & \longrightarrow & 0 \\ & & \| & & T\downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & U & \longrightarrow & 0 \end{array}$$

commutative. The three-lemma and the open mapping theorem imply that $T$ must be an isomorphism [10, 1.5]. An exact sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow U \longrightarrow 0$ is said to split and $X$ is said to be trivial if it is equivalent to the trivial sequence $0 \longrightarrow Y \longrightarrow Y \oplus U \longrightarrow U \longrightarrow 0$.

A quasi-linear map $F : U \to Y$ where $U$ and $Y$ are Banach spaces is a homogeneous map such that

$$\|F(u+z) - F(u) - F(z)\| \leq k(\|u\| + \|z\|)$$

for some constant $k$ and all $u,z \in U$. For a quasi-linear map $F : U \to Y$, there corresponds a twisted sum $Y \oplus_F U$ by endowing the product space $Y \times U$ with the quasi-norm $\left\|(y,z)\right\| = \left\|y - F(z)\right\| + \|z\|$. The subspace $\{(y,0) : y \in Y\}$ of $Y \oplus_F U$ is isometric to $Y$ and the corresponding quotient $(Y \oplus_F U)/Y$ is isomorphic to $U$. Conversely, for every twisted sum of $Y$ and $U$ there is a quasi-linear map $F : U \to Y$ such that $X$ is equivalent to $Y \oplus_F U$ [10,1.5]. Two quasi-linear maps $F$ and $G$ of a Banach space $U$ into a Banach space $Y$ are said to be equivalent if the corresponding twisted sums$Y \oplus_F U$ and $Y \oplus_G U$ are equivalent. If the quasi-linear map $F : U \longrightarrow Y$, acting between two Banach spaces $U$ and $Y$, is zero-linear, that is $F$ satisfies

$$\left\|F\left(\Sigma_{i=1}^{n} u_i\right) - \Sigma_{i=1}^{n} F\left(u_i\right)\right\| \le k\left(\Sigma_{i=1}^{n} \|u_i\|\right).$$

for some constant $k$, where $u_1, u_2, ...u_n$ are finitely many elements of $U$, then the twisted sum $Y \oplus_F U$ is locally convex [10, 1.6.e]. We denote by $Ext(U,Y)$ the space of all equivalence classes of locally convex twisted sums of $Y$ and $U$. Thus $Ext(U,Y) = 0$ means that all locally convex twisted sums of $Y$ and $U$ are equivalent to the direct sum $Y \oplus U$.

Given a family $\mathcal{E}$ of finite dimensional Banach spaces, a Banach space $X$ is said to contain $\mathcal{E}$ uniformly complemented if there exists a constant $c$ such that for every $E \in \mathcal{E}$, there is a $c$-complemented subspace $A$ of $X$ which is $c$-isomorphic to $E$. It is clear that $X$ contains $\mathcal{E}$ uniformly complemented if and only if its second dual $X^{**}$does. A Banach space $X$ is said to be $\lambda$-locally $\mathcal{E}$ (or locally $\mathcal{E}$) if there exists a constant $\lambda > 1$ such that every finite dimensional subspace $A$ of $X$ is contained in a finite dimensional subspace $B$ of $X$ such that

$$d_{BM}(B,E) = \inf\{\|T\| \left\|T^{-1}\right\|; T : X \longrightarrow Y \text{ is an isomorphism of } X \text{ onto} Y\} < \lambda$$

for some $E \in \mathcal{E}$ [6].

We say that a Banach space $X$ is $\lambda$-colocally$\mathcal{E}$ (or colocally$\mathcal{E}$) if there exists a constant $\lambda > 1$ such that every finite dimensional quotient $A$ of $X$ is a quotient of another finite dimensional quotient $B$ of $X$ satisfying $d_{BM}(B,E) < \lambda$ for some $E \in \mathcal{E}$ [17].

The locality of a family is a very useful tool to determine the existence of nontrivial twisted sums of certain Banach spaces, in fact, Cabello and Castillo proved

**Theorem 1** $[6, Theorem2]$ *Let $\mathcal{E}$ be a family of finite dimensional Banach spaces and let $W$ be a Banach space containing $\mathcal{E}$ uniformly complemented. If $Y$ is a Banach space complemented in its bidual such that $Ext(W,Y) = 0$, then $Ext(Z,Y) = 0$ for every Banach space $Z$ locally $\mathcal{E}$.*

Jebreen et all proved the corresponding version for the colocallity of Banach spaces as follows

**Theorem 2** [18, *Theorem*1.7] *Let $\mathcal{E}$ be a family of finite dimensional Banach spaces and let $W$ be a Banach space containing $\mathcal{E}$ uniformly complemented. If $Y$ is a Banach space such that $Ext(Y,W) = 0$, then $Ext(Y,Z) = 0$ for every Banach space $Z$ complemented in its bidual and colocally $\mathcal{E}$.*

The triviality of all twisted sums of two Banach spaces is inherited by their complemented subspaces.

**Proposition 3** [5, *Lemma*3], [17, *Proposition*2.3] *Let $X$, $A_1$ and $A_2$ be Banach spaces such that $X = A_1 \oplus A_2$. Then for any Banach space $U$*
(*i*) $Ext(U,X) = 0$ *if and only if* $Ext(U,A_i) = 0$ *for* $i = 1,2$.
(*ii*) $Ext(X,U) = 0$ *if and only if* $Ext(A_i,U) = 0$ *for* $i = 1,2$.

Throughout this paper $\mathbb{K}$ denotes the scalar field; either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and $c_{00}$ denotes the vector space of all finitely supported sequences in $\mathbb{K}$, that is $c_{00} := \{(\alpha_n) : \alpha_n \in K,\ n \in N$ and $\exists N \in \mathbb{N} : n = 0, \forall n > N\}$.

## 2. Twisted sums with James spaces.

In 1951, Robert C. James provided the first example of a non-reflexive Banach space isomorphic to its second dual, called the James space $J_2$ [15 ]. Edelstein and Mityagin were the first to observe that it can be generalized to an arbitrary $p > 1$ as they defined

$$\|x\|_{J_p} = \sup\left\{\left(\sum_{i=1}^{k}\left|a_{n_j} - a_{n_{j+1}}\right|^p\right)^{\frac{1}{p}} : k, n_1, n_2, ..., n_{k+1} \in \mathbb{N}, n_1 < n_2 < ... < n_{k+1}\right\}$$

and the Banach space $\left\{x = (x_n)_{n\in\mathbb{N}} \in c_0 : \|x\|_{J_p} < \infty\right\}$ is called the p$^{\mathbf{th}}$ James Space $J_p$. It can be seen that the completion of $c_{00}$ with respect to this norm is $J_p$ [12]. Moreover, Edelstein and Mityagin showed that James' proof of the quasi-reflexivity of $J_2$, the original James space, can be carried out to $J_p$ for every $p > 1$ [12]. The proof can not, however, work for $p = 1$ because $J_1$ is isometrically isomorphic to $\ell_1$. Bird et all [4,2.3] proved that $J_p$ contains a complemented copy of $\ell_p$, for $1 < p < \infty$.

**Proposition 4** *(i) $Ext(J_p, \ell_1) \neq 0$, that is $Ext(J_p, J_1) \neq 0, 1 < p < \infty$.*
*(ii) $Ext(J_p, J_q) \neq 0, 1 < p, q < \infty$.*

**Proof.** (i) Note that $c_0$ is finitely represented in $J_p$ [3, Theorem 1.1], that is for each $\epsilon > 0$, and each finite-dimensional subspace $E$ of $c_0$, there exists a subspace $F$ of $J_p$, depending on $E$ such that there is an isomorphism $T$ on $E$ onto $F$ satisfying $\|T\|\left\|T^{-1}\right\| < 1 + \epsilon$ [3]. Hence $J_p$ contains $\{\ell_\infty^n\}_{n=1}^\infty$ uniformly complemented. Since $Ext(c_0, \ell_1) \neq 0$ [6, Theorem 5.1], then the result can be deduced by Theorem 1.

(ii) For $1 < p < \infty$ , $J_p$ contains a complemented copy of $\ell_p$ [4,2.3], and $\ell_p$ contains $\left\{l_p^n\right\}$ uniformly complemented [23, II.5.9], then $J_p$ contains $\left\{l_p^n\right\}$ uniformly complemented. Hence $Ext\left(\ell_p, \ell_q\right) \neq 0$, where $1 < p, q < \infty$ [7, Section 5], implies that $Ext\left(J_p, \ell_q\right) \neq 0$ by Theorem 1 and $Ext\left(\ell_p, J_q\right) \neq 0$ by Theorem 2. Therefore $Ext\left(J_p, J_q\right) \neq 0$ by Proposition 3. ∎

The Schreier space $S_1$ was first considered by Schreier in 1930 [26], in order to provide an example of a weakly null sequence without Cesaro summable subsequence. A variation of this idea gave rise to the construction of the Schreier spaces [2], [8]. Bird and Laustsen generalized the concept of a Schreier space from one Schreier space, corresponding to the $\ell_1$-norm, to a whole family, one for each $p \geq 2$ , corresponding to the $\ell_p$-norms as follows:

$$\|x\|_{Z_p} = \sup_A \left(\Sigma_{j \in A} \left|x_j\right|^p\right)^{\frac{1}{p}}$$

where $x = (x_n)_{n \in \mathbb{N}}$ and the supremum is taken over all admissible subsets of $\mathbb{N}$, which are defined as the finite subsets $A = \{n_1, n_2, \ldots, n_k\}$ of $N$ such that $k \leq n_1 < n_2 < \ldots < n_k$. The subspace $\left\{x = (x_n)_{n \in \mathbb{N}} \in c_0 : \|x\|_{Z_p} < \infty\right\}$ of $\mathbb{K}^{\mathbb{N}}$ is a Banach space called the unrestricted $\mathrm{p}^{\mathbf{th}}$ Schreier Space $Z_p$**.** The completion of $c_{00}$ with respect to$\|x\|_{Z_p}$ is the restricted $\mathrm{p}^{\mathbf{th}}$ Schreier Space $S_p$ [4,3.2,3.6].

In 2010, Bird and Laustsen create a new family of Banach spaces, the James–Schreier spaces, by amalgamating the two important classical Banach spaces: James' quasi-reflexive Banach space and Schreier's Banach space and they proved that these spaces are counterexamples to the Banach–Saks property and that most of the results about the James space as a Banach algebra carry over to the new spaces; see [4] for details. For $1 \leq p < \infty$, they defined the following norm:

$$\|x\|_{W_p} = \sup_A \left(\sum_{i=1}^{k} \left|x_{n_j} - x_{n_{j+1}}\right|^p\right)^{\frac{1}{p}}$$

where the supremum is taken over all permissible subsets of $\mathbb{N}$, which are defined as the finite subsets $A = \{n_1, n_2, \ldots, n_{k+1}\}$ of $N$ such that $k \leq n_1 < n_2 < \ldots < n_{k+1}$. The subspace $Z_p = \left\{x = (x_n)_{n \in \mathbb{N}} \in c_0 : \|x\|_{W_p} < \infty\right\}$ of $\mathbb{K}^{\mathbb{N}}$ is a Banach space called the unrestricted $\mathrm{p}^{\mathbf{th}}$ James-Schreier Space $W_p$. The completion of $c_{00}$ with respect to$\|x\|_{W_p}$ is the restricted $\mathrm{p}^{\mathbf{th}}$James-Schreier Space $V_p$ [4,4.2,4.8].

**Proposition 5** *Let* $U \in \left\{\ell_q, S_q, V_q, Z_q, W_q, V_q^{**}, 1 < q < \infty\right\}$. *Then*

$(i)$ $Ext(J_p, U) \neq 0, 1 < p < \infty.$
$(ii)$ $Ext(U, J_p) \neq 0, 1 < p < \infty.$

**Proof.** Since $Ext(\ell_p, U) \neq 0$, and $Ext(U, \ell_p) \neq 0, 1 < p < \infty$, where $U$ is either $\ell_q, S_q$ ,$V_q$ ,$Z_q$, $W_q$, or $V_q^{**}$ by [19, Proposition 2.4], and $\ell_p$ is complemented in $J_p$

then $Ext(J_p, U) \neq 0$ and $Ext(U, J_p) \neq 0$ by Proposition 3. ■

The original James space $J_2$ has an additional twisted sums due to $Ext(U, \ell_2) \neq 0$, where $U$ is either $c_0, S_1, V_1, Z_1, W_1$ or $V_1^{**}$[19, Proposition 2.3] which gives

**Proposition 6** $Ext(U, J_2) \neq 0$ *where* $U \in \left\{c_0, S_1, V_1, Z_1, W_1, V_1^{**}\right\}$.

**3. Twisted sums of the Tsirelson's space and the AD space.**

A Banach space X is said to be asymptotic $\ell_p$, $1 \leq p \leq \infty$ with respect to a basis $(e_i)_{i=1}^{\infty}$, if there exists a constant $C \geq 1$ (the asymptotic constant), such that $(x_i)_{i=1}^{n}$ is $C$-equivalent to the unit vector basis of $\ell_p^n$, i.e. for any n-tuple of scalars $a = (a_1, \ldots, a_N)$,

$$\frac{1}{c}\|a\|_p \leq \left\|\Sigma_{i=1}^{n} a_i x_i\right\| \leq c\|a\|_p$$

In 1974, Tsirelson [27] constructed the first example of a reflexive Banach space with unconditional basis that is asymptotically $\ell_\infty$ but has no embedded copy of $c_0$ or $\ell_p\,(1 \leq p < \infty)$. He constructed a convex, weakly compact subset $V$ of $c_0$ with the following properties:

(i) For all $n \in N, e_n \in V$,

(ii) If $f = (f_n) \in V$ and $g = (g_n)$ is such that $|g_n| \leq |f_n|$ for all $n \in N$, then $g \in V$,

(iii) If $f_1, \cdots, f_n \in V$ are such that $n \leq f_1 < \cdots < f_n$, then $\frac{1}{2}(f_1 + \cdots + f_n) \in V$,

(iv) For all $x \in V$ there exists $n \in N$ such that $2P_n(x) \in V$.

Tsirelson's space is the linear span of the set $V$ with the norm that makes $V$ be the unit ball. Figiel and Johnson [14] constructed the conjugate of Tsirelson example, known as the $T$ space. It is defined as the closure of the finitely supported sequences with the norm $\|x\|_T := \lim \|x\|_m$, where the norms $\|x\|_m$are defined inductively:

$$\|x\|_0 = \|x\|_{c_0}\ , \|x\|_{m+1} = \{\max \|x\|_m, \frac{1}{2}\max[\sum_{j=1}^{n} \left\|E_j x\right\|_m]\}$$

and the inner max is taken over all choices of consecutive finite sets $\{E_j\}_{j=1}^{n}$, $n \leq E_1 < E_2 < \ldots < E_n$ [14]. The original Tsirelson space is denoted in the literature by $T^*$.

Recall that $\{l_\infty^n\} \subseteq c_0$, for each $n \in \mathbb{N}$, hence $c_0$ is locally $\{l_\infty^n\}$.

**Proposition 7** $Ext(T^*, U) \neq 0$, *where* $U \in \left\{\ell_q, S_q, Z_q, W_q, V_q, V_q^{**}, J_q, q = 1, 2\right\}$.

**Proof.** Since $T^*$ is asymptotically $\ell_\infty$ then $T^*$ contains $\{l_\infty^n\}$ uniformly complemented. Hence $Ext(c_0, \ell_1) \neq 0$ [7, Theorem 5.1], and $Ext(c_0, \ell_2) \neq 0$ [6, 4.2] imply $Ext(T^*, \ell_1) \neq 0$, and $Ext(T^*, \ell_2) \neq 0$ by Theorem 1. For $U \in \left\{\ell_q, S_q, Z_q, W_q, V_q, V_q^{**}, J_q, q = 1, 2\right\}$, $U$ contains $\left\{\ell_q^n\right\}$uniformly complemented Theorem 2. ■

Argyros and Deliyanni constructed two examples of asymptotic $\ell_1$ Banach spaces, both are of the type of Tsirelson's [1]. The second, we shall denote it by $AD$, does not contain any unconditional basic sequence.

**Proposition 8** *(i)* $Ext(U,T) \neq 0$, *and* $Ext(U,AD) \neq 0$ *where* $U \in \left\{c_0, S_p, V_p, Z_p, W_p, V_p^{**}, 1 \leq p < \infty\right\}$.
*(ii)* $Ext(T^*,T) \neq 0$, *and* $Ext(T^*,AD) \neq 0$

**Proof.** (i) By [19, Proposition 2.2], $Ext(U,\ell_1) \neq 0$ where $U \in \left\{c_0, S_p, V_p, Z_p, W_p, V_p^{**}, 1 \leq p < \infty\right\}$. Since $T$ and $AD$ are asymptotically $\ell_1$, then $T$ and $AD$ contain $\left\{\ell_1^n\right\}$ uniformly complemented. The result follows by Theorem 2.

(ii) Since $Ext(T^*,\ell_1) \neq 0$ by the previous proposition, and since the spaces $T$ and $AD$ contain $\left\{\ell_1^n\right\}$ uniformly complemented, the result follows by Theorem 2. ∎

## 4. Twisted sums of the Johnson-Lindenstrauss space.

Johnson and Lindenstrauss [20] constructed a nontrivial twisted sum of $c_0$ and a Hilbert space (necessarily non-separable), called the Johnson-Lindenstrauss space $JL$. It is defined to be the completion of the linear span of $c_0 \cup \{\chi_i : i \in I\}$ in $\ell_\infty$ with respect to the norm:

$$\left\|y = x + \Sigma_{j=1}^{k} a_{i(j)}\chi_{i(j)}\right\| = \max\left\{\left\|y\right\|_\infty, \left\|(a_i)_{i\in I}\right\|_{\ell_2(I)}\right\}$$

$x \in c_0$ , $a_{i(j)}$ are scalars, and $\chi_i$ is the characteristic function of $A_i$, $\{A_i\}_{i\in I}$ is an almost disjoint uncountable family of infinite subsets of $\mathbb{N}$. They proved that $JL/c_0$ is isomorphic to some $\ell_2(I)$ and since $\ell_1$ is projective, the dual sequence $0 \longrightarrow \ell_2(I) \longrightarrow JL^* \longrightarrow \ell_1 \longrightarrow 0$ of the exact sequence $0 \longrightarrow c_0 \longrightarrow JL \longrightarrow \ell_2(I) \longrightarrow 0$ splits. That is, $JL^* = \ell_1 \oplus \ell_2(I)$.

**Proposition 9** *(i)* $Ext(JL^*,U) \neq 0$, *where* $U \in \left\{\ell_1, S_1, Z_1, W_1, V_1, V_1^{**}, T, AD\right\}$.
*(ii)* $Ext(U,JL^*) \neq 0$, *where* $U \in \left\{T^*, c_0, S_p, Z_p, W_p, V_p, V_p^{**}, 1 \leq p < \infty\right\}$.

**Proof.** (i) Since $JL^* = \ell_1 \oplus \ell_2(I)$ and $\ell_2$ is complemented in $\ell_2(I)$, then $\ell_2$ is complemented in $JL^*$. By [6, 4.3] $Ext(\ell_2,\ell_1) \neq 0$, hence $Ext(JL^*,\ell_1) \neq 0$, which leads to $Ext(JL^*,T) \neq 0$, $Ext(JL^*,AD) \neq 0$, $Ext(JL^*,S_1) \neq 0$ and $Ext(JL^*,Z_1) \neq 0$ by Theorem 2. By proposition 3, the result follows.

(ii) Since $\ell_1$ is complemented in $JL^*$, then the result can be concluded using $Ext(c_0,\ell_1) \neq 0$ and a similar argument to that used in (i). ∎

In [17, Theorem 2.2] it has been proved that for any Banach spaces $U$ and $Y, Ext(Y,U^*) = 0$ if and only if $Ext(U,Y^*) = 0$. Therefore it is immediate that

**Corollary 10** $Ext(JL,U^*) \neq 0$, *where* $U \in \left\{T^*, c_0, S_p, Z_p, W_p, V_p, V_p^{**}, 1 \leq p < \infty\right\}$.

## 5. Twisted sums of the James tree space.

The James tree space JT was introduced by James in [15]. It was the first example of a separable dual Banach space that contains no copy of $\ell_1$ though it has a non separable dual. Moreover, James proved that $JT$ does not contain a subspace isomorphic to $c_0$ or $\ell_1$. The space $JT$ is defined to be the completion of the space of finite sequences over the dyadic tree $\Delta$ with respect to the norm

$$\|x\| = \sup_{n\in N} \sup_{S_1,...,S_n} \left[\Sigma_{i=1}^{n} \left(\Sigma_{\alpha\in S_i} x_\alpha\right)^2\right]^{1/2}$$

where the supremum is taken over all finite sets of pair wise disjoint segments of $\Delta$.

**Proposition 11** *(i)* $Ext(JT,U) \neq 0$, *where* $U \in \left\{\ell_1, S_1, Z_1, W_1, V_1, V_1^{**}, T, AD, JL^*\right\}$.
*(ii)* $Ext(JT,U) \neq 0$, *where* $U \in \left\{\ell_2, S_2, Z_2, W_2, or V_2, V_2^{**}, JL^*\right\}$.

**Proof.** (i) Fetter de Buen [13, 2.b.8, 3.a.7] proved that $c_0$ is finitely represented in the James space $J$. Since $JT$ contains $J$, then $JT$ contains $\{\ell_\infty^n\}_{n=1}^\infty$ uniformly which implies that it contains $\{\ell_\infty^n\}_{n=1}^\infty$ uniformly complemented. Since $Ext(c_0,\ell_1) \neq 0$, then by Theorem 1 we have $Ext(JT,\ell_1) \neq 0$. Applying Theorem 2 gives $Ext(JT,T) \neq 0, Ext(JT,AD) \neq 0, Ext(JT,S_1) \neq 0$ and $Ext(JT,Z_1) \neq 0$. The result follows by Proposition 3.

(ii) The proof is by using $Ext(c_0,\ell_2) \neq 0$ and applying Theorems 1, 2, and Proposition 3. ■

**Proposition 12** $Ext(U,\mathcal{B}) \neq 0$, *where* $\mathcal{B}$ *is the predual of* $JT$, *and* $Ext(U,JT^*) \neq 0$, *where* $U \in \left\{\ell_p, S_p, Z_p, W_p, V_p, V_p^{**}, 1 \leq p < \infty\right\}$.

**Proof.** The predual $B$ of $JT$ contains $\left\{\ell_1^n\right\}_{n=1}^\infty$ uniformly complemented [18, Lemma 2.4], and hence so does $JT^*$. It has been proved in [19, Proposition 2.2] that $Ext(U,\ell_1) \neq 0$, where $U$ is either $\ell_p, S_p, V_p, Z_p, W_p$, or $V_p^{**}$, so by Theorem 2 we get the result. ■

**6. Singular Twisted sums with C(K) spaces.**

A continuous linear operator $T : E \to F$ between two Banach spaces is called strictly singular if it fails to be invertible on any infinite dimensional closed subspace of $E$. We say that a quasi-linear map $F : U \to Y$ is strictly singular if it has no trivial restriction to any infinite dimensional subspace of $U$. A quasi-linear map $F : U \to Y$ is strictly singular if and only if the quotient map $Q : Y \oplus_F U \to U$ is a strictly singular operator, that is the restriction of $Q$ to any infinite dimensional subspace of $Y \oplus_F U$ is not an isomorphism [11, Lemma 1]. In this case we say that the twisted sum $Y \oplus_F U$ is singular.

**Theorem 13** *There are singular twisted sums* $U \oplus_F C[0,1]$, *where* $U \in \{T, T^*, AD, JT\}$.

**Proof.** For every $U \in \{T, T^*, AD, JT\}$, $U$ is separable and have no copy of $\ell_1$, hence there is an exact sequence

$$0 \to C[0,1] \xrightarrow{j} X \xrightarrow{Q} U \to 0$$

with Q strictly singular [7, 2.3]. Hence the corresponding quasi-linear maps $F : U \to C[0,1]$ is strictly singular. ■

Recall that if $N \in \mathbb{N}$, the space $C(\omega^N)$ is isomorphic to $c_0$, and so, by Sobczyk's Theorem, for any separable Banach space $U$, we have $Ext(U, C(\omega^N)) = 0$. The extension constant $\pi_N(U)$ is the least constant such that if

$$0 \to C(\omega^N) \xrightarrow{j} X \xrightarrow{Q} U \to 0$$

is an exact sequence and $\varepsilon > 0$, then there is a linear operator $P : X \to C(\omega^N)$ with $Pj = i_{C(\omega^N)}$ and $\|P\| \leq \pi_N(U) + \varepsilon$ [7, Section 3]. It is proved that $\pi_N(U) \leq 2N + 1$, for every $N \in \mathbb{N}$ [7, Theorem 3.1].

**Theorem 14** $Ext(T, C(\omega^\omega)) = 0$.

**Proof.** Recall that $T^*$ is a separable Banach space with summable Szlenk index [21, Proposition 6.7]. Hence $Ext(T, C(\omega^\omega)) = 0$ by [7, 4.4], which implies that $Sup_N \pi_N(U) < \infty$ by [7,4.1]. ■

## 7. Pelczynski's property (u).

Two infinite-dimensional Banach spaces $X$ and $Y$ are totally incomparable if no closed, infinite-dimensional subspace of $X$ is isomorphic to a subspace of $Y$. Since $\left\{J_p, S_q\right\}$ and $\left\{J_p, V_q\right\}$ are totally incomparable for $p, q \geq 1$ [3,5.9]; we can deduce immediately that any twisted sum that extends $J_p$ can not be isomorphic to $S_q$ or $V_q$, $p, q \geq 1$. But we can know more about the twisted sums of certain spaces using Pełczynski's property $(u)$. A Banach space $X$ has Pełczynski's property $(u)$ if for every weak Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $X$, there is a sequence $(y_n)_{n\in\mathbb{N}}$ in $X$ such that for every bounded functional on $X$ we have

$$\sum_{n=1}^{\infty} \left|\langle y_n, f \rangle\right| < \infty \text{ and } \left\langle x_n - \sum_{i=1}^{n} y_j, f \right\rangle \to 0 \text{ as } n \to \infty$$

We will show that for any Banach space $U$, every twisted sum that extends $J_2$ or the $p^{th}$-James–Schreier space $V_p$ $(p > 1)$ does not have the Pełczynski's property $(u)$. For this purpose we need the Pełczynski's Theorem:

**Theorem 15 (22)** *(i) Every Banach space with an unconditional basis has Pełczynski's property (u).*

*(ii) Every closed subspace of a Banach space with Pełczynski's property (u) has Pełczynski's*
*property (u).*

*(iii) The James space $J_2$ does not have Pełczynski's property (u).*

**Theorem 16** *The twisted sum that extends $V_p (p > 1)$ or $J_2$ does not have the*

*Pełczynski's property $(u)$, and hence has no unconditional basis.*

**Proof.** Let $X \in Ext\left(U, V_p\right)$, then $V_p$ is isomorphic to a closed subspace of $X$. Since $V_p$ does not have the Pełczynski's property $(u)$ [4, Theorem 6.3] then $X$ does not have Pełczynski's property $(u)$, by $(ii)$ of Pelczynski theorem. Hence $X$ has no unconditional basis. The case for $X \in Ext(U, j_2)$ can be proved similarly by using (iii) of the previous theorem. ■

**Data Availability**

The readers who are interested may contact the author for more informations.

**Conflicts of Interest**

The author declares no conflict of interest.

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